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Some classes of projected dynamical systems in Banach spaces and variational inequalities

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Abstract We introduce some Projected Dynamical Systems based on metric and generalized Projection Operator in a strictly convex and smooth Banach Space. Then we prove that critical points of these systems coincide with the solution of a Variational Inequality.

Keywords Projected dynamical systems · Variational inequalities · Equilibrium points · Critical points · Alber's decomposition theorem

1 Introduction

The authors Cojocaru, Daniele, Isac, Nagurney, and Raciti (see [5–7, 13, 14, 17]) initiated the systematic study of Projected Dynamical Systems (PDS) on infinite dimensional Hilbert Spaces and applied this new concept to the study of Evolutionary Variational Inequality (EVI) by proving that critical points of the PDS coincides with the solutions (equilibrium points) of EVI.

To be analytically precise the PDS is the differential equation given by:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lim_{\lambda \to 0} \frac{P_C(x - \lambda F(x)) - x}{\lambda} = P_{T_C(x)}(-F(x)),$$

where P_C denotes the standard projection on a closed convex subset C of an Hilbert Space.

This result has a remarkable importance from both applied and theoretic point of view. In a natural way, the problem arises to study if a similar concept can be stated in a Banach

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Space proving also an equivalence theorem. In this paper, we solve this problem using new effective concepts of projection and show the desired equivalence theorems. Precisely we introduce the Metric Projected Dynamical System (m-PDS), and the Generalized Projected Dynamical System (g-PDS), using for m-PDS the minimum norm projection operator, and for g-PDS the Generalized Projection operator based on Lyapunov Function. With the aid of Alber's decomposition Theorem we are able to present interesting proprieties of these new concepts and point out open problems which can be the object of new researches.

The main result of our paper, namely the equivalence between the critical points of the m-PDS and g-PDS introduced and the equilibrium points of a variational inequalities, can be applied to the study of variational inequalities in the Banach Spaces with a particular emphasis to evolutionary variational inequalities which express traffic, market or financial equilibria.

Also these topics will be object of further researches. Some additional results can be found in [11].

2 The duality mappings J and J^*

We denote by X a Banach space with dual space X^* and by $\|.\|$ and $\|.\|_*$ the respective norms. We denote also the duality pairing between X^* and X by $< f, x >$ for $f \in X^*$ and $x \in X$, $< x, f >$ the duality pairing between X and X^* for $f \in X^*$ and $x \in X$.

We define the duality mapping J: $X \to X^*$ by

$$J(x) = \{f \in X^* :< f, x >= \|f\|_*^2 = \|x\|^2\}, \quad \forall x \in X.$$

In the same manner we have the duality mapping J^*: $X^* \to X$ defined by:

$$J^*(f) = \{x \in X :< x, f >= \|x\|^2 = \|f\|_*^2\}, \quad \forall f \in X^*.$$

The existence of J and J^* is a corollary of the Hahn–Banach analytic form (see for instance [4]).

Remark 1 X is a Hilbert space if and only if J is linear. If we identify X with its topological dual, then $J = Id_X = J^*$.

Example 1 If $X = L^p(\Omega, \mathbb{R})$ with $1 < p < \infty$ then

$$J(x) = \|x\|^{2-p}|x|^{p-1}sgn(x)$$

and

$$J^*(x) = \|x\|^{\frac{p-2}{p-1}}|x|^{\frac{1}{1-p}}sgn(x)$$

where $sgn(x) = \chi_{[x>0]} - \chi_{[x<0]}$.
This result could be usefully applied to Time Dependent Traffic Equilibria problems (see [9]).

Now we recall two definitions we need in the sequel.

Definition 1 (*see* [10]) A space $(X, \|.\|)$ is strictly convex if

$$\forall x \in X, \quad \forall y \in X : \|x\| = \|y\| = 1, \quad x \neq y \Rightarrow \|tx + (1-t)y\| < 1, \quad \forall t \in]0, 1[.$$

Let us denote by $S(X) = \{x \in X : \|x\| = 1\}$.

Definition 2 (*see* [10]) A Banach space X is said to be smooth at $x_0 \in S(X)$ whenever there exists a unique $f \in S(X^*)$ such that $f(x_0) = 1$. If X is smooth at each point of $S(X)$ then we say that X is smooth.

From [10] we have also the following characterization criteria: a Banach space $(X, \|.\|)$ is smooth if and only if the norm $\|.\|$ admits a Gâteaux derivative in each direction.

Remark 2 Hilbert spaces and L^p spaces $(1 < p < \infty)$ are reflexive, strictly convex and smooth.

From [3] we know that if we have X reflexive, strictly convex and smooth then J, J^* are one-to-one single-valued operators and $J^{-1} = J^*$. More precisely we have:

- X is reflexive if and only if J is surjective;
- X is smooth if and only if J is single-valued;
- X is strictly convex if and only if J is injective.

3 Metric and generalized projection operators

Besides the notion of projection operator in Hilbert space, it is possible to give an effective projection operator definition in a more general framework. Let us recall the following definition of metric projection operator (for more details see for instance [18]).

Definition 3 (*see* [18]) Let X be a Banach space and C a closed convex subset of X. We call the metric projection operator from X on C the set valued mapping $\pi(C|.)$: $X \to C$ defined by

$$x \to \pi(C|x) = \{y \in C : \|x - y\| = d_C(x)\}$$

where $d_C(x) = \inf_{z \in C} \|x - z\|$.

Note that for $x \in C$, $\pi(C|x)$ is the set of optimal solution of the following minimization problem:

$$\inf_{y \in C} \|x - y\|^2. \tag{1}$$

From now on and unless otherwise stated, we make the followings assumptions: X Banach space, reflexive, strictly convex, and smooth.

Then these additional assumptions ensure that $\pi(C|.) = P_C(.)$ is single valued and P_C is called the best approximate operator. Moreover we have the following characterization of $P_C(x)$:

$$\bar{x} = P_C(x) \Leftrightarrow < J(x - \bar{x}), y - \bar{x} > \leq 0, \quad \forall y \in C. \tag{2}$$

As an extension of what we have on Hilbert spaces, (2) is called the basic variational principle for P_C in X. This characterization plays a fundamental role for our application.

Another possibility to generalize the notion of projection is to use, as done by Alber in [1], the *Lyapunov function*.

The Lyapunov function is the strictly convex function in y, $V(x, y)$ given by:

$$V(x, y) := \|x\|^2 - 2 < J(x), y > + \|y\|^2.$$

We remark that if C is a closed convex subset of X and if $x \in C$ then the problem

$$\min_{y \in C} V(x, y)$$

is uniquely solvable (apply for instance [4], Corollary III.20) , then we can give the following definition:

Definition 4 (*see* [1] *or* [18]) We call generalized projection of x on C the following value:

$$\Pi_C(x) := arg \min_{y \in C} V(x, y).$$

Remark 3 (see [1])

- The operator Π_C: $X \to C \subset X$ is the identity on C, i.e., for every $x \in C$, $\Pi_C(x) = x$.
- In a Hilbert space, $V(x, y) = \|x - y\|^2$, Π_C coincides with the projection operator P_C.

As stated in [2] we have the following characterization of $\Pi_C(x)$.

Lemma 1 *Assume that C is a closed convex subset of X, then:*

$$\hat{x} = \Pi_C(x) \Leftrightarrow < J(x) - J(\hat{x}), y - \hat{x} > \leq 0, \quad \forall y \in C. \tag{3}$$

Here again the variational characterization plays a fundamental role for our application.

From Corollary 1, p. 22 [10], we know that if X is reflexive then:

$$X \text{ strictly convex } \Leftrightarrow X^* \text{ smooth,}$$
$$X \text{ smooth } \Leftrightarrow X^* \text{ strictly convex.}$$

Definition (3) applies also to X^* and to convex and closed subset $\Gamma \in X^*$, and we have the following variational principle:

$$\bar{f} = P_\Gamma(f) \Leftrightarrow < J^*(f - \bar{f}), g - \bar{f} > \leq 0, \quad \forall g \in \Gamma. \tag{4}$$

We can introduce also the Lyapunov function on $X^* \times X^*$:

$$V^*(f, g) = \|f\|_* - 2 < J^*(f), g > + \|g\|_*$$

and then the following definition

Definition 5 We call generalized projection of f on $\Gamma \in X^*$, the following value:

$$\Pi_\Gamma(f) := \arg \min_{g \in \Gamma} V^*(f, g).$$

We have the following variational principle:

$$\hat{f} = \Pi_\Gamma(f) \Leftrightarrow < J^*(f) - J^*(\hat{f}), g - \hat{f} > \leq 0, \quad \forall g \in \Gamma. \tag{5}$$

4 Tangent cone, normal cone, and relative interior

We recall for readers utility the following basic definitions and properties.

Definition 6 Let $C \subset X$ be a convex, we call Tangent cone to C in x the set given by:

$$T_C(x) = \overline{\bigcup_{\lambda > 0} \lambda(C - x)}.$$

Remark 4 The Definition 6 is valid if C is convex. The general definition of the tangent cone for nonconvex set is given by:

$$T_C(x) = \limsup_{\lambda \to 0} \frac{1}{\lambda}(C - x)$$

Definition 7 Let $C \subset X$ be a convex, we call Normal cone to C in x the set given by:

$$N_C(x) = \{\xi \in X^*, < \xi, y - x > \leq 0, \forall y \in C\}.$$

Definition 8 Let M be a cone of X, the polar set of M, noted M^0 is defined by:

$$M^0 = \{\xi \in X^*, < \xi, x > \leq 0, \forall x \in M\}.$$

If **X is reflexive**, then the following relationships hold:

$$\begin{aligned} (T_C(x))^0 &= N_C(x), \quad \forall x \in C, \\ (N_C(x))^0 &= T_C(x), \quad \forall x \in C. \end{aligned} \tag{6}$$

T_C and N_C are always closed and if C is nonempty and convex they are nonempty and convex. These cones are used to introduce the relative interior (see [12]).

Definition 9 Let $C \subset X$ be a convex. We call the relative interior of C the following set:

$$\mathrm{ri}(C) = \{x \in C : T_C(x) = X\}.$$

Definition 10 Let $C \subset X$ be a convex. We call the relative boundary of C the following set:

$$\mathrm{rb}(C) = C \setminus \mathrm{ri}(C).$$

Proposition 1 (Proposition 2.2 in [8]) *Let us assume that X is a reflexive Banach space and $C \subset X$ a convex. If $x \in C$ we have:*

$$x \in \mathrm{ri}(C) \Leftrightarrow N_C(x) = \{0_{X^*}\}.$$

Proof If $T_C(x) = X$ then we have:

$$N_C(x) = \{\xi \in X^* :< \xi, x > \leq 0, \forall x \in X\}$$

so we get $\forall x \in X, < \xi, x > \leq 0$, and $< \xi, x > \geq 0$ so we can deduce that $\xi = 0_{X^*}$.

On the other side if $N_C(x) = \{0_{X^*}\}$ then using the polarity we get

$$T_C(x) = \{\xi \in X :< \xi, 0_{X^*} > \leq 0\} = X$$

and by definition $x \in \mathrm{ri}(C)$. □

These notions reveal to be very useful in infinite dimensions because many convex sets used in Variational analysis have a topological interior void and a relative interior nonvoid (see [8]).

5 Projected dynamical systems in Banach spaces

In [6] the authors introduced in the framework of Hilbert spaces the Operator

$$\Pi_C(x, -F(x)) = \lim_{\lambda \to 0} \frac{P_C(x - \lambda F(x)) - x}{\lambda} = P_{T_C(x)}(-F(x)) \tag{7}$$

and applied this new operator to the study of a class of differential equations called Projected Dynamical Equations and to the study of Variational Inequalities (VI).

Our aim is to introduce in the framework of Reflexive, smooth, and strictly convex Banach space an operator with a lot of proprieties of $\Pi_C(x, -F(x))$ and apply this concept to VI in Banach Space. We propose the following new definitions:

Definition 11 We call the Metric Projected Dynamical System operator, the operator

$$\Lambda_C^m \colon C \times X^* \to X$$

defined by setting:

$$\Lambda_C^m(x, h) = P_{T_C(x)}(J^*(h)).$$

So we can define as done in [6,16] the differential equation with a discontinuous right-hand side.

Definition 12 We call M-Projected Dynamical Equation (m-PrDE), the discontinuous right-hand side differential equation given by:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \Lambda_C^m(x, -F(x)) = P_{T_C(x)}(J^*(-F(x))). \tag{8}$$

Consequently the associated Cauchy problem is given by:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \Lambda_C^m(x, -F(x)) = P_{T_C(x)}(J^*(-F(x))), \quad x(0) = x_0 \in C. \tag{9}$$

A Metric Projected Dynamical System (m-PDS) is the dynamical system given by the set of trajectories of a m-PrDE.

Definition 13 We call the g-PDS operator, the operator

$$\Lambda_C^g \colon C \times X^* \to X$$

defined by setting:

$$\Lambda_C^g(x, h) = \Pi_{T_C(x)}(J^*(h)).$$

Definition 14 We call g-PrDE, the discontinuous right-hand side differential equation given by:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \Lambda_C^g(x, -F(x)) = \Pi_{T_C(x)}(J^*(-F(x))). \tag{10}$$

The associated Cauchy problem is given by:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \Lambda_C^g(x, -F(x)) = \Pi_{T_C(x)}(J^*(-F(x))), \quad x(0) = x_0 \in C. \tag{11}$$

A Generalized Projected Dynamical System (g-PDS) is the dynamical system given by the set of trajectories of a g-PrDE

In a Hilbert Space X, in which we choose as dual realization X, both (8) and (10) are equal to (7).

6 Decomposition theorem

In this section, we provide a result demonstrated by Alber ([2]) which generalize the Moreau Theorem (see [15]).

Theorem 1 ([2], Theorem 2.4) *Assume that X is a real reflexive strictly convex and smooth Banach space, and K a nonempty, closed and convex cone of X then:* $\forall x \in X$ *and* $\forall f \in X^*$ *the decompositions*

$$\begin{aligned} x &= P_K(x) + J^* \Pi_{K^0} J(x) \quad \textit{and} \quad < \Pi_{K^0} J(x), P_K(x) >= 0, \\ f &= P_{K^0}(f) + J \Pi_K J^*(f) \quad \textit{and} \quad < P_{K^0}(f), \Pi_K J^*(f) >= 0 \end{aligned} \tag{12}$$

hold.

Remark 5 If X is a Hilbert space identified with its dual, the decomposition $x = P_K(x) + J^* \Pi_{K^0} J(x)$ reduces to $x = P_K(x) + P_{K^0}(x)$.

Corollary 1 *For each* $v \in X^*$ *we have:*

$$\Lambda_C^m(x, v) = J^*(v) - J^* \Pi_{N_C(x)}(v). \tag{13}$$

Proof From Theorem 1 with $K = T_C(x)$ and $K^0 = N_C(x)$, we get:

$$J^*(v) = P_{T_C(x)}(J^*(v)) + J^* \Pi_{N_C(x)} J(J^*(v))$$

as $J J^* = Id_{X^*}$ and $P_{T_C(x)}(J^*(v)) = \Lambda_C^m(x, v)$ we deduce immediately the result.

Corollary 2 *For each* $v \in X^*$ *we have:*

$$\Lambda_C^g(x, v) = J^*(v - P_{N_C(x)}(v)). \tag{14}$$

Proof From Theorem 1 with $K = T_C(x)$ and $K^0 = N_C(x)$, we get:

$$v = P_{N_C(x)}(v) + J \Pi_{T_C(x)}(J^*(v)).$$

As $\Pi_{T_C(x)}(J^*(v)) = \Lambda_C^g(x, v)$ we deduce immediately the result.

7 Equivalence theorems

We present the main results of our work, namely we show that the critical points of $m - PrDS$ (8) and $g - PrDS$ (10) are the equilibrium points of following variational inequality:

$$x \in C : < F(x), v - x > \geq 0, \quad \forall v \in C, \tag{15}$$

where F: $C \to X^*$.

Let us recall some results regarding the existence of equilibria for (15). There are two standard approaches to the existence of equilibria, namely, with and without a monotonicity requirement.

We shall employ the following definitions.

Definition 15 (*see* [9]) Let E be a real topological vector space, $C \subset E$ convex. Then F: $C \to E^*$ is said to be:

(1) pseudomonotone iff, for all $x, y \in C$, $< F(x), y - x >\geq 0 \Rightarrow < F(y), x - y >\leq 0$;

(2) hemicontinous iff, for all $y \in C$, the function $\xi \to < F(\xi), y - \xi >$ is upper semicontinous on C;
(3) hemicontinous along line segments iff, for all $x, y \in C$, the function $\xi \to < F(\xi), y - x >$ is upper semicontinous on the line segment $[x, y]$.

Then we have the following result.

Theorem 2 (see [9]) *Let E be a real topological vector space, and let $C \subseteq E$ be convex and nonempty. Let $F: C \to E^*$ be given such that:*

(1) *there exist $A \subseteq C$, compact, and $B \subseteq C$ compact, convex such that, for every $x \in C \setminus A$, there exists $y \in B$ with $< F(x), y - x >< 0$;*
either (2) or (3) below holds:
(2) *F is hemicontinous;*
(3) *F is pseudomonotone and hemicontinous along line segments.*

Then, there exists $\bar{x} \in A$ such that $< F(\bar{x}), y - \bar{x} >\geq 0$, for all $y \in C$.

Theorem 3 *Assume that the hypotheses of Theorems (1) and (2) hold. Then each equilibrium point of (15) is a critical point of (8) and, if (8) admits critical points then they are equilibrium points of (15).*

Proof Let x^* be a solution of (15), since J is bijective there exists an unique $u_{x^*} \in X$ such that $-F(x^*) = J(u_{x^*})$.
So we have

$$< -J(u_{x^*}), x - x^* >\geq 0, \quad \forall x \in C$$

and then

$$< -J(u_{x^*}), \lambda(x - x^*) >\geq 0, \quad \forall x \in C\ \forall \lambda > 0,$$

which is equivalent to write:

$$< J(u_{x^*} - 0_X), y - 0_X >\leq 0, \quad \forall y \in T_C(x^*).$$

So using the variational principle (2) for $P_{T_C(x^*)}$ we get

$$P_{T_C(x^*)}(u_{x^*}) = 0_X = P_{T_C(x^*)}(J^*(-F(x^*)))$$

and we deduce that x^* is a critical point of (8).

Now suppose that x^* is a critical point of (8).

We have $P_{T_C(x^*)}(J^*(-F(x^*))) = 0_X$ and by Corollary 1 we get

$$J^*(-F(x^*)) = J^* \Pi_{N_C(x^*)}(-F(x^*))$$

as $(J^*)^{-1} = J$ we get

$$-F(x^*) = \Pi_{N_C(x^*)}(-F(x^*)).$$

If $x^* \in ri(C)$: then $N_C(x^*) = 0_{X^*}$ so we get:

$$\Pi_{N_C(x^*)}(w) = \Pi_{0_{X^*}}(w) = 0_{X^*} = -F(x^*),\ \forall w \in X^*$$

so we deduce that x^* is solution of (15).

If $x^* \in \mathrm{rb}(C)$ and $J^*(-F(x^*)) \notin T_C(x^*)$ we get $N_C(x^*) \neq 0_{X^*}$ and taking into account that $-F(x^*) = \Pi_{N_C(x^*)}(-F(x^*))$, we deduce that $-F(x^*) \in N_C(x^*)$ and so, using the definition of $N_C(x^*)$ we obtain

$$< F(x^*), x - x^* > \geq 0, \quad \forall x \in C,$$

which implies that x^* is solution of (15).

If $x^* \in \mathrm{rb}(C)$ and $J^*(-F(x^*)) \in T_C(x^*)$ we derive immediately

$$P_{T_C(x^*)}(J^*(-F(x^*))) = 0_X = J^*(-F(x^*))$$

but J^* is an isometry and so $-F(x^*) = 0_{X^*}$. Then again x^* is solution of (15). □

Remark 6 In the previous proof, it is possible to avoid the use of $\mathrm{ri}(C)$, but this notion permits to have an easier approach to geometrical aspects of the theorem.

Theorem 4 *Assume that the hypotheses of Theorems* (1) *and* (2) *hold. Then each equilibrium point of* (15) *is a critical point of* (10) *and, if* (10) *admits critical points then they are equilibrium points of* (15).

Proof Let x^* be a solution of (15), since J is bijective there exists an unique $u_{x^*} \in X$ such that $-F(x^*) = J(u_{x^*})$.
So we have

$$< -J(u_{x^*}), x - x^* > \geq 0, \quad \forall x \in C$$

and then

$$< -J(u_{x^*}), \lambda(x - x^*) > \geq 0, \quad \forall x \in C\ \forall \lambda > 0,$$

which is equivalent to write:

$$< J(u_{x^*}) - J(0_X), y - 0_X > \leq 0, \quad \forall y \in T_C(x^*).$$

So using the variational principle (3) for $\Pi_{T_C(x^*)}$ we get

$$\Pi_{T_C(x^*)}(u_{x^*}) = 0_X = \Pi_{T_C(x^*)}(J^*(-F(x^*)))$$

from which we deduce that x^* is a critical point of (10).

Now suppose that x^* is a critical point of (10).

$\Pi_{T_C(x^*)}(J^*(-F(x^*))) = 0_X$ and by Corollary 2 we get

$$J^*(-F(x^*) - P_{N_C(x^*)}(-F(x^*))) = 0_X \Leftrightarrow -F(x^*) = P_{N_C(x^*)}(-F(x^*)).$$

If $F(x^*) = 0_{X^*}$ then (15) is trivially verified. Now we suppose that $F(x^*) \neq 0_{X^*}$. Then as $-F(x^*) = P_{N_C(x^*)}(-F(x^*))$ we get $-F(x^*) \in N_C(x^*)$ which means

$$< -F(x^*), y - x^* > \leq 0, \quad \forall y \in C$$

and this is exactly (15). □

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